1 Functions of Several Variables **2021-22**

Notation

The subject of this course is the study of functions $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$. The elements of $\mathbb{R}^n$, for $n \geq 2$, will be called *vectors* so, if $m > 1$, $\mathbf{f}$ will be said to be a *vector-valued* function of *several variables*. If $m = 1$ we will say f is *scalar-valued*. Note my use of bold face for vectors, in lectures I will have to underline the symbol.

All vectors should be written vertically and we will follow the so called 'contravariant convention' of using superscript labels for coordinates of vectors. So $\mathbf{v} \in \mathbb{R}^n$ is written in coordinates as

$$\mathbf{v} = \begin{pmatrix} v^1 \\ v^2 \\ \vdots \\ \vdots \\ v^n \end{pmatrix}$$

where each coordinate $v^i \in \mathbb{R}$. Though I might write vertical vectors in lectures it would be wasteful of space in lecture notes so I will type this as

$$\mathbf{v} = \left(v^1, v^2, \cdots, \cdots, v^n\right)^T .$$

Even in lectures I may not have the room for vertical vectors and may write them horizontally, but **without** the superscript T.

For a sequence of vectors we will use subscripts $\mathbf{v}_1, \mathbf{v}_2$, etc. We may then put these vectors into a matrix

$$(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m) = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{v}_1, & \mathbf{v}_2, & ..., & \mathbf{v}_m \\ \downarrow & \downarrow & & \downarrow \end{pmatrix} = \begin{pmatrix} v_1^1 & v_2^1 & v_3^1 & \cdots & v_m^1 \\ v_1^2 & v_2^2 & v_3^2 & & v_m^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ v_1^n & v_2^n & v_3^n & \cdots & v_m^n \end{pmatrix} .$$

I would recommend remembering the labeling in this matrix as a mnemonic for remembering the super and sub-script labeling.

An important sequence of vectors is the *standard* basis for $\mathbb{R}^n$, denoted by $\mathbf{e}_i$, $1 \leq i \leq n$. So $\mathbf{e}_i$ has 1 in the i-th coordinate, 0 elsewhere. Then $\mathbf{v} \in \mathbb{R}^n$ can be written as

$$\mathbf{v} = \sum_{i=1}^{n} v^i \mathbf{e}_i,$$

with each $v^i \in \mathbb{R}$ for $1 \leq i \leq n$.

Proofs

Many proofs of results for vector-valued functions of several variables are based on the ideas seen in the proof of the equivalent result for scalar-valued functions of one variable, i.e. results seen in MATH20101 or MATH20111. Such proofs will not given. I will say they are not given because no new ideas are required.

But lots of proofs are given; this is not a 'calculus' course concerned only with calculation and application, but an 'analysis' course concerned with *why* a result holds. The proofs we give will make essential use of the fact that in $\mathbb{R}^n$ there are many paths to a given point whereas in $\mathbb{R}$ there are far fewer. (Such a difference also exists between $\mathbb{C}$-analysis and $\mathbb{R}$-analysis.)

The methods we develop also lead to alternative proofs of results which also have 'proofs with no new ideas', such as the Limit Laws for functions of several variables. These alternative proofs will be given.

Unless stated otherwise all proofs given in the notes should be learnt.

1.1 Limits of vector-valued functions of several variables.

Most of the results on limits of real-valued, or what will be called in this course **scalar-valued**, functions of **one** real variable generalise in a straightforward way to limits of **vector-valued** functions $\mathbf{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ of **several variables**.

For $\mathbf{x} \in \mathbb{R}^n$ define $|\mathbf{x}|^2 = \sum_{i=1}^n (x^i)^2$. Perhaps the most useful observations are that

- $|x^i| \leq |\mathbf{x}|$ for $1 \leq i \leq n$, and
- the **triangle inequality**, $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

The triangle inequality can be applied $n-1$ times to prove

- $|\mathbf{x}| \leq \sum_{i=1}^n |x^i|$.

Definition 1 *Given a point $\mathbf{a} \in \mathbb{R}^n$ and a real number $\delta > 0$ the open ball*

$$B_\delta(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{a}| < \delta\}$$

*is call a **neighbourhood** or δ **- neighbourhood** of $\mathbf{a}$. We say that $B_\delta(\mathbf{a})$ has **centre** $\mathbf{a}$ and **radius** δ.*

The set

$$B'_\delta(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^n : 0 < |\mathbf{x} - \mathbf{a}| < \delta\} = B_\delta(\mathbf{a}) \backslash \{\mathbf{a}\}$$

*is called a **punctured** or **deleted** neighbourhood of $\mathbf{a}$.*

Definition 2 *Assume that $\mathbf{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a function with domain A containing a deleted neighbourhood of $\mathbf{a} \in \mathbb{R}^n$. Then $\mathbf{f}$ has the **limit** $\mathbf{b} \in \mathbb{R}^m$ **at** $\mathbf{a}$ iff*

$$\forall \varepsilon > 0,\, \exists \delta > 0 : \forall \mathbf{x} \in \mathbb{R}^n,\, 0 < |\mathbf{x} - \mathbf{a}| < \delta \implies |\mathbf{f}(\mathbf{x}) - \mathbf{b}| < \varepsilon.$$

In this case we write $\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ or $\mathbf{f}(\mathbf{x}) \to \mathbf{b}$ as $\mathbf{x} \to \mathbf{a}$.

Note 1 δ is chosen sufficiently small such that $B'_\delta(\mathbf{a}) \subseteq A$ so $\mathbf{f}$ is defined on all points $\mathbf{x} : 0 < |\mathbf{x} - \mathbf{a}| < \delta$. The function $\mathbf{f}$ need not be defined at $\mathbf{a}$.

Note 2 The definition could be written as

$$\forall \varepsilon > 0,\, \exists \delta > 0 : \forall \mathbf{x} \in \mathbb{R}^n,\, \mathbf{x} \in B'_\delta(\mathbf{a}) \implies \mathbf{f}(\mathbf{x}) \in B_\varepsilon(\mathbf{b}).$$

The following is IMPORTANT.

Theorem 3 *Assume that $\mathbf{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a function with domain A containing a deleted neighbourhood of $\mathbf{a} \in \mathbb{R}^n$. If $\mathbf{f}$ has a limit as $\mathbf{x} \to \mathbf{a}$ then the limit is unique.*

Proof No new ideas; this can be proved in exactly the same way as in the single variable case. (See Appendix) ■

1.2 Scalar-valued examples

Stress Definition 2 may be identical to that for scalar-valued functions of one variable, but the verification of Definition 2 in any example can be far more complicated.

Example 4 *By verifying the* ε - δ *definition show that the scalar-valued* $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y)^T \mapsto xy$ *has limit* 6 *at* $\mathbf{a} = (3, 2)^T$.

Solution *Rough Work.* Assume $\mathbf{x}$ satisfies $0 < |\mathbf{x} - \mathbf{a}| < \delta$ with δ to be chosen. In particular, individually $|x - 3| < \delta$ and $|y - 2| < \delta$. Consider $f(\mathbf{x}) - 6 = xy - 6$ and rearrange to try to use the fact that $x - 3$ and $y - 2$ are small. For example

$$\begin{aligned} xy - 6 &= (x-3)(y-2) + 2x + 3y - 12 \\ &= (x-3)(y-2) + 2(x-3) + 3(y-2). \end{aligned}$$

Then, by the triangle inequality,

$$|f(\mathbf{x}) - 6| \le |x-3|\,|y-2| + 2\,|x-3| + 3\,|y-2| \le \delta^2 + 5\delta$$

since $|x - 3| < \delta$ and $|y - 2| < \delta$. It is unnecessarily complicated to have a δ^2 factor so demand that $\delta \le 1$ in which case $|f(\mathbf{x}) - 6| < \delta + 5\delta = 6\delta$. *End of Rough work.*

Proof Let $\varepsilon > 0$ be given. Choose $\delta = \min(1, \varepsilon/6)$. Assume that $0 < |\mathbf{x} - \mathbf{a}| < \delta$. Then, for such $\mathbf{x}$, the argument above shows that

$$|f(\mathbf{x}) - 6| < 6\delta \le 6\left(\frac{\varepsilon}{6}\right) = \varepsilon.$$

Hence we have verified the definition of $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = 6$. ■

This could be repeated at a general point $\mathbf{a} \in \mathbb{R}^2$.

Example 5 *By verifying the* ε - δ *definition show that the scalar-valued* $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y)^T \mapsto xy$ *has limit* ab *at* $\mathbf{a} = (a, b)^T$.

Solution *in Problems Class.*

Rough work Assume $\mathbf{x}$ satisfies $0 < |\mathbf{x} - \mathbf{a}| < \delta$ with δ to be chosen. In particular $|x - a| < \delta$ and $|y - b| < \delta$. Then

$$f(\mathbf{x}) - ab = xy - ab = (x-a)(y-b) + b(x-a) + a(x-b).$$

So, by the triangle inequality,

$$\begin{aligned}|f(\mathbf{x}) - ab| &\leq |x-a|\,|y-b| + |b|\,|x-a| + |a|\,|x-b| \\ &< \delta^2 + (|b| + |a|)\,\delta \\ &\leq (1 + |b| + |a|)\,\delta,\end{aligned}$$

on demanding $\delta \leq 1$.

End of Rough work.

Proof Let $\varepsilon > 0$ be given. Choose $\delta = \min(1, \varepsilon/(1 + |a| + |b|))$. Assume that $0 < |\mathbf{x} - \mathbf{a}| < \delta$. Then, for such $\mathbf{x}$, the argument above shows that

$$|f(\mathbf{x}) - 5| < (1 + |b| + |a|)\,\delta \leq (1 + |b| + |a|)\left(\frac{\varepsilon}{(1 + |b| + |a|)}\right) = \varepsilon.$$

Hence we have verified the definition of $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = ab$. ■

1.3 Vector-valued examples

Example 6 *By verifying the ε-δ definition show that the vector-valued function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$, given by*

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ xy \end{pmatrix},$$

has limit $(5,6)^T$ at $\mathbf{a} = (3,2)^T$.

Solution With $\mathbf{b} = (5,6)^T$ consider

$$|\mathbf{f}(\mathbf{x}) - \mathbf{b}|^2 = \left|\begin{pmatrix} x+y-5 \\ xy-6 \end{pmatrix}\right|^2 = (x+y-5)^2 + (xy-6)^2.$$

Would now wish to bound this in terms of $|\mathbf{x} - \mathbf{a}|$. Or, in terms of $|x-3|$ and $|y-2|$ since both are less than or equal to $|\mathbf{x} - \mathbf{a}|$.

This looks complicated, and I certainly don't have time to complete it in lectures (but see Appendix). There must be a simpler approach. ■

1.4 From vector-valued to scalar-valued

Suppose that $\mathbf{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Then for each $\mathbf{x} \in A$,

$$\mathbf{f}(\mathbf{x}) = \left(f^1(\mathbf{x}), f^2(\mathbf{x}), ..., f^m(\mathbf{x})\right)^T$$

for real numbers $f^1(\mathbf{x}), f^2(\mathbf{x}), ..., f^m(\mathbf{x})$. This defines m *scalar-valued* functions $f^i : A \to \mathbb{R}$ for $1 \leq i \leq m$ which are called the *components* or *component functions* of the vector-valued function $\mathbf{f}$.

The next result is important, as seen in the frequency of its use. In short, vector-valued functions have a limit if, and only if, each scalar-valued component function has a limit.

Proposition 7 *Assume that* $\mathbf{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *is a function with domain* A *containing a deleted neighbourhood of* $\mathbf{a} \in \mathbb{R}^n$*. Then* $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$ *if, and only if,* $\lim_{\mathbf{x}\to\mathbf{a}} f^i(\mathbf{x}) = b^i$ *for* $1 \leq i \leq m$.

Proof ($\Rightarrow$) Assume that $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$. Let $\varepsilon > 0$ be given. Then, by definition, there exists $\delta > 0$ such that

$$0 < |\mathbf{x} - \mathbf{a}| < \delta \implies |\mathbf{f}(\mathbf{x}) - \mathbf{b}| < \varepsilon.$$

But for any $1 \leq i \leq m$ we have $|f^i(\mathbf{x}) - b^i| \leq |\mathbf{f}(\mathbf{x}) - \mathbf{b}|$. Hence

$$0 < |\mathbf{x} - \mathbf{a}| < \delta \implies \left|f^i(\mathbf{x}) - b^i\right| < \varepsilon.$$

Thus we have verified the definition of $\lim_{\mathbf{x}\to\mathbf{a}} f^i(\mathbf{x}) = b^i$.

($\Leftarrow$) Assume $\lim_{\mathbf{x}\to\mathbf{a}} f^i(\mathbf{x}) = b^i$ for all $1 \leq i \leq m$. Let $\varepsilon > 0$ be given. Then for each i there exists $\delta^i > 0$ such that

$$0 < |\mathbf{x} - \mathbf{a}| < \delta^i \implies \left|f^i(\mathbf{x}) - b^i\right| < \varepsilon/\sqrt{m}. \tag{1}$$

Let $\delta = \min\left\{\delta^i : 1 \leq i \leq m\right\}$. Then

$$\begin{aligned} 0 < |\mathbf{x} - \mathbf{a}| < \delta &\implies \forall\, 1 \leq i \leq m,\ 0 < |\mathbf{x} - \mathbf{a}| < \delta \leq \delta^i \\ &\implies \forall\, 1 \leq i \leq m,\ \left|f^i(\mathbf{x}) - b^i\right| < \varepsilon/\sqrt{m}, \end{aligned}$$

by (1). Hence

$$|\mathbf{f}(\mathbf{x}) - \mathbf{b}|^2 = \sum_{i=1}^m \left|f^i(\mathbf{x}) - b^i\right|^2 < \sum_{i=1}^m \left(\frac{\varepsilon}{\sqrt{m}}\right)^2 = \varepsilon^2.$$

That is

$$0 < |\mathbf{x} - \mathbf{a}| < \delta \implies |\mathbf{f}(\mathbf{x}) - \mathbf{b}| < \varepsilon.$$

Therefore we have verified the ε - δ definition of $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$. ■

Exercise for student. Rewrite the proof using the fact that $|\mathbf{x}| \leq \sum_{i=1}^{n} |x^i|$ in place of the triangle inequality.

Example 8 *Using Proposition 7 show that* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$, *given by*

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ xy \end{pmatrix},$$

has limit $(5,6)^T$ *at* $\mathbf{a} = (3,2)^T$.

Solution This is Example 6 again, though this time we are not required to verify the definition. Instead, Proposition 7 says it suffices to prove that

i. $f : \mathbb{R}^2 \to \mathbb{R}, (x,y)^T \mapsto x + y$ has limit 5 at $(3,2)^T$,

ii. $g : \mathbb{R}^2 \to \mathbb{R}, (x,y)^T \mapsto xy$ has limit 6 at $(3,2)^T$.

Part i is the first question on the Problem Sheet while Part ii was Example 4 above. Thus the result follows. ■

1.5 Sandwich Rule

For limits which are zero we have

Lemma 9 ***Sandwich Rule***. *Let* $\mathbf{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *and* $h : A \subseteq \mathbb{R}^n \to \mathbb{R}^{\geq 0}$. *Assume* $0 \leq |\mathbf{f}(\mathbf{x})| \leq h(\mathbf{x})$ *for* $\mathbf{x}$ *in a deleted neighbourhood of* $\mathbf{a}$. *If* $\lim_{\mathbf{x}\to\mathbf{a}} h(\mathbf{x}) = 0$ *then* $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{0}$.

Proof By assumption there exists $\delta_1 > 0$ such that if $0 < |\mathbf{x} - \mathbf{a}| < \delta_1$ then $0 \leq |\mathbf{f}(\mathbf{x})| \leq h(\mathbf{x})$.

Let $\varepsilon > 0$ be given. Then $\lim_{\mathbf{x}\to\mathbf{a}} h(\mathbf{x}) = 0$ implies there exists $\delta_2 > 0$ such that if $0 < |\mathbf{x} - \mathbf{a}| < \delta_2$ then $|h(\mathbf{x}) - 0| < \varepsilon$, i.e. $h(\mathbf{x}) < \varepsilon$ since $h \geq 0$.

Let $\delta = \min(\delta_1, \delta_2)$ and assume $\mathbf{x}$ satisfies $0 < |\mathbf{x} - \mathbf{a}| < \delta$. For such $\mathbf{x}$ we have $|\mathbf{f}(\mathbf{x})| \leq h(\mathbf{x}) < \varepsilon$. This can be rewritten as $|\mathbf{f}(\mathbf{x}) - \mathbf{0}| < \varepsilon$. So we have shown that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |\mathbf{x} - \mathbf{a}| < \delta$ then $|\mathbf{f}(\mathbf{x}) - \mathbf{0}| < \varepsilon$, the definition of $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{0}$. ■

For a scalar-valued example of the use of the Sandwich Rule we have

Example 10 *Let*

$$f(\mathbf{x}) = \frac{x^4 - y^6}{x^2 + y^2} \quad \textit{for} \quad \mathbf{x} = (x, y)^T \neq \mathbf{0}.$$

Show that $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x}) = 0$.

Solution Bound the function using $|\mathbf{x}|^2 = x^2 + y^2$,

$$|f(\mathbf{x})| = \left|\frac{x^4 - y^6}{x^2 + y^2}\right| = \frac{|x^4 - y^6|}{|\mathbf{x}|^2} \leq \frac{|x|^4 + |y|^6}{|\mathbf{x}|^2},$$

by the triangle inequality on the numerator. Next use the fact that $|x|, |y| \leq |\mathbf{x}|$ to get

$$|f(\mathbf{x})| \leq \frac{|x|^4 + |y|^6}{|\mathbf{x}|^2} \leq \frac{|\mathbf{x}|^4 + |\mathbf{x}|^6}{|\mathbf{x}|^2} = |\mathbf{x}|^2 + |\mathbf{x}|^4 \to 0$$

as $\mathbf{x} \to \mathbf{0}$. Hence, by the Sandwich Rule with $h(\mathbf{x}) = |\mathbf{x}|^2 + |\mathbf{x}|^4$, we have $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x}) = 0$. ■

1.6 Directional Limits

Definition 11 *Assume that* $\mathbf{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *is a vector-valued function with domain* A *containing a deleted neighbourhood of* $\mathbf{a} \in \mathbb{R}^n$*. The* ***directional limit of* $\mathbf{f}$ *at* $\mathbf{a}$ *from the direction* $\mathbf{v}$** *is*

$$\lim_{t\to 0+} \mathbf{f}(\mathbf{a}+t\mathbf{v}),$$

if it exists.

Note If this limit exists it does not depend on the size of $\mathbf{v}$, only it's direction, so $\mathbf{v}$ does not have to be a unit vector. Also, it is important that this is a **one-sided** limit, for the limit approaching the point $\mathbf{a}$ along a direction $\mathbf{v}$ may be different from approaching $\mathbf{a}$ from along $-\mathbf{v}$, the 'other side'.

Example 12 *Let* $f : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \mathbb{R}$ *be given by*

$$f(\mathbf{x}) = \frac{x^4 - y^2}{x^4 + y^2} \quad \textit{where } \mathbf{x} = (x, y)^T.$$

Find the directional limits at $\mathbf{0}$ *for all directions.*

Solution Let $\mathbf{v} = (u, v)^T \in \mathbb{R}^2$ be a non-zero vector. Then

$$f(\mathbf{0} + t\mathbf{v}) = \frac{t^4u^4 - t^2v^2}{t^4u^4 + t^2v^2} = \frac{t^2u^4 - v^2}{t^2u^4 + v^2} \to \frac{-v^2}{v^2} = -1$$

as $t \to 0+$, *as long as* $v \neq 0$.

If $v = 0$ then $\mathbf{v} = (u, 0)^T$ and

$$f(\mathbf{0} + t\mathbf{v}) = \frac{t^4u^4}{t^4u^4} = 1.$$

In conclusion, the directional limit is -1 for all directions except for $\mathbf{v} = (u, 0)^T$ when the directional limit is 1. ■

Be very aware of the $v = 0$ case, most students fail to see it.

If the limit exists we have

Lemma 13 *Assume that* $\mathbf{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *is a vector-valued function with domain* A *containing a deleted neighbourhood of* $\mathbf{a} \in \mathbb{R}^n$. *Assume that* $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} \in \mathbb{R}^m$. *Then, for any non-zero vector* $\mathbf{v} \in \mathbb{R}^n$, *the directional limit of* $\mathbf{f}$ *at* $\mathbf{a}$ *from the direction* $\mathbf{v}$ *exists and further*

$$\lim_{t\to 0+} \mathbf{f}(\mathbf{a}+t\mathbf{v}) = \mathbf{b}.$$

Proof No new ideas, this is an example of a limit of a composition as seen last year. See Problem Sheet ■

In fact, the proof of Lemma 13 shows that if $\mathbf{f}$ has limit $\mathbf{b}$ at $\mathbf{a}$ then the *two-sided* limit satisfies $\lim_{t\to 0} \mathbf{f}(\mathbf{a}+t\mathbf{v}) = \mathbf{b}$. See the appendix for a brief discussion of the limit as $t \to 0-$.

In part, this lemma says

$$\boxed{\text{limit exists} \implies \text{all directional limits exist.}}$$

The converse if false - see problem sheet for examples. The reason the converse fails is that directional limits only approach a point along straight lines. There are many other paths leading to any given point, as seen in the next section.

So, remember

$$\boxed{\forall\, \mathbf{v},\ \lim_{t\to 0+} \mathbf{f}(\mathbf{a} + t\mathbf{v}) = \mathbf{b} \not\Longrightarrow \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}.}$$

The contrapositive of Lemma 13 is the **useful**

$$\boxed{\begin{array}{c} \exists\, \mathbf{v}\,(\neq \mathbf{0}) \in \mathbb{R}^n : \lim\limits_{t\to 0+} \mathbf{f}(\mathbf{a}{+}t\mathbf{v}) \text{ does not exist} \\ \text{OR } \exists\, \mathbf{v}_1, \mathbf{v}_2 \neq \mathbf{0} : \lim\limits_{t\to 0+} \mathbf{f}\,(\mathbf{a}{+}t\mathbf{v}_1) \neq \lim\limits_{t\to 0+} \mathbf{f}\,(\mathbf{a}{+}t\mathbf{v}_2) \\ \implies \lim\limits_{\mathbf{x}\to\mathbf{a}} \mathbf{f}\,(\mathbf{x}) \text{ does NOT exist.} \end{array}}$$

That is, we can show that $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})$ does **not** exist by either finding a direction in which the limit does not exist or by finding two different directions with different limits.

Example 14 *Show that neither*

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{x^2y+y}{x^3+y^3} \quad \textit{nor} \quad \lim_{\mathbf{x}\to\mathbf{0}} \frac{x^4-y^2}{x^4+y^2}$$

exist.

Solution For the first limit $f(t\mathbf{e}_2) = 1/t^2$ which has no limit as $t \to 0$.

For the second limit the result directly from Example 12 where we found two different directional limits at $\mathbf{0}$. ■

1.7 Limits along curves

Definition 11 concerns the composition of $\mathbf{f}$ with a line $t \mapsto \mathbf{a} + t\mathbf{v}$. The line can be replaced by a curve in $\mathbb{R}^n$, the image of some $\mathbf{g} : (0,\eta) \to \mathbb{R}^n$. (A curve, or path, will be defined more carefully later in the course.)

Definition 15 *If* $\mathbf{g} : (0,\eta) \to A\backslash\{\mathbf{a}\} \subseteq \mathbb{R}^n$ *is a vector-valued function of one variable such that* $\lim_{t\to 0+} \mathbf{g}\,(t) = \mathbf{a}$ *then*

$$\lim_{t\to 0+} \mathbf{f}(\mathbf{g}\,(t))$$

is called the limit of $\mathbf{f}$ ***at*** $\mathbf{a}$ ***along*** $\mathbf{g}$*, if it exists.*

Example 10 revisited *Find the limit at* $t = 0$ *of*

$$f(\mathbf{x}) = \frac{x^4 - y^6}{x^2 + y^2}$$

for $\mathbf{x} = (x, y)^T \neq \mathbf{0}$, *along the curve* $\mathbf{g}(t) = (t^2, t^3)^T$, $t \geq 0$.

Solution

$$f(\mathbf{g}(t)) = \frac{t^8 - t^{18}}{t^4 + t^6} = \frac{t^4 - t^{14}}{1 + t^2}.$$

Then,

$$\lim_{t \to 0+} f(\mathbf{g}(t)) = \lim_{t \to 0+} \frac{t^4 - t^{14}}{1 + t^2} = \frac{\lim_{t \to 0+} (t^4 - t^{14})}{\lim_{t \to 0+} (1 + t^2)},$$

by the Quotient Rule for limits. We are allowed to use the Quotient Rule since both limits on the right hand side exist and $\lim_{t \to 0+} (1 + t^2) = 1 \neq 0$. Thus

$$\lim_{t \to 0+} f(\mathbf{g}(t)) = \frac{0}{1} = 0.$$

■

Similarly Lemma 13 can be generalised:

Lemma 16 *Assume* $\mathbf{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *where* A *contains a deleted neighbourhood of* $\mathbf{a} \in \mathbb{R}^n$. *Assume that* $\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} \in \mathbb{R}^m$ *exists. Then, for any vector-valued function of one variable* $\mathbf{g} : (0, \eta) \to A \backslash \{\mathbf{a}\}$ *such that* $\lim_{t \to 0+} \mathbf{g}(t) = \mathbf{a}$, *we have*

$$\lim_{t \to 0+} \mathbf{f}(\mathbf{g}(t)) = \mathbf{b}.$$

Proof No new ideas, it is another example of a limit of a composition. ■

Note We look at the limit of $\mathbf{f}$ at $\mathbf{a}$ since $\mathbf{f}$ may not be defined at $\mathbf{a}$. For this reason we exclude the possibility that $\mathbf{g}$ takes the value $\mathbf{a}$ by demanding that the image of $\mathbf{g}$ lies in $A \backslash \{\mathbf{a}\}$.

In part this result says

$$\boxed{\text{limit exists} \implies \text{limit exists along any path.}}$$

Again **the converse is not true**.

Lemma 13 is a special case of Lemma 16 with $\mathbf{g}(t) = \mathbf{a} + t\mathbf{v}$.

The contrapositive of Lemma 16 is the useful

$$\boxed{\begin{array}{c} \exists\, \mathbf{g} : (0,\eta) \to \mathbb{R}^n \setminus \{\mathbf{a}\} : \lim_{t\to 0+} \mathbf{f}(\mathbf{g}(t)) \text{ does not exist} \\ \text{OR } \exists\ \mathbf{g}_1, \mathbf{g}_2 : \lim_{t\to 0+} \mathbf{f}(\mathbf{g}_1(t)) \neq \lim_{t\to 0+} \mathbf{f}(\mathbf{g}_2(t)) \\ \implies \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) \text{ does NOT exist.} \end{array}}$$

In the next example $f : \mathbb{R}^2 \to \mathbb{R}$ is a function that takes non-zero values on non-zero points on the parabola $y = x^2$, and 0 elsewhere.

Example 17 *Show that*

$$f(\mathbf{x}) = \begin{cases} 1 \textit{ if } \mathbf{x} = (x, x^2)^T,\ x \neq 0 \\ 0 \textit{ otherwise} \end{cases}$$

has all their directional limits at $\mathbf{x} = \mathbf{0}$ *but no limit at* $\mathbf{x} = \mathbf{0}$.

Solution *Problems Class.* On any straight line $t\mathbf{v} = (tu, tv)^T$ the function is non-zero only when $tv = (tu)^2$, i.e. $t = v/u^2$, *provided* $u \neq 0$.

If $v \leq 0$ then there is no $t > 0$ satisfying $t = v/u^2$ so $f(t\mathbf{v}) = 0$ for all $t > 0$ and thus $\lim_{t\to 0+} f(t\mathbf{v}) = 0$.

If $\nu > 0$ then $f(t\mathbf{v}) = 0$ for all $0 < t < v/u^2$ and so, again, $\lim_{t\to 0+} f(t\mathbf{v}) = 0$.

Finally, if $u = 0$ then the line is $(0, tv)$ and the function is always 0 on the line hence the limit is 0.

Therefore in all directions the directional limit is 0.

If we look at the curve $\mathbf{g}(t) = (t, t^2)^T$ we find $f(\mathbf{g}(t)) = 1$ and so $\lim_{t\to 0+} f(\mathbf{g}(t)) = 1$.

If $\lim_{\mathbf{x}\to\mathbf{0}} f(\mathbf{x})$ exists then the limits along all straight lines and curves would exist and *have the same value.* We have seen that this is not the case, hence the limit does not exist at $\mathbf{x} = \mathbf{0}$. ∎

In the Appendix the limit in this example is shown not to exist directly by using the definition of a limit.

1.8 Continuity of vector-valued functions of several variables

Definition 18 *Assume that* $\mathbf{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *is a function with domain* A *containing a neighbourhood of* $\mathbf{a} \in \mathbb{R}^n$*. Then* $\mathbf{f}$ *is* ***continuous at*** $\mathbf{a}$ *iff* $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$*.*

This has an ε - δ version:

$$\forall\varepsilon > 0,\, \exists\,\delta > 0 : \forall\mathbf{x} \in A, |\mathbf{x} - \mathbf{a}| < \delta \implies |\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a})| < \varepsilon,$$

or

$$\forall\varepsilon > 0,\, \exists\delta > 0 : \forall\mathbf{x} \in \mathbb{R}^n,\, \mathbf{x} \in B_\delta(\mathbf{a}) \implies \mathbf{f}(\mathbf{x}) \in B_\varepsilon(\mathbf{b})\,.$$

The idea of being continuous at *a* point is not of as much use as being continuous in a *set* of points. We will be **assuming** that any such set will contain a neighbourhood of **any** point within it. Such sets are called *open sets.*

Definition 19 *A subset* $U \subseteq \mathbb{R}^n$ *is an* ***open subset*** *if, for each* $\mathbf{a} \in U$*,* U *contains a neighbourhood of* $\mathbf{a}$*. That is,*

$$\forall\mathbf{a} \in U, \exists r > 0 : B_r(\mathbf{a}) \subseteq U.$$

Note that $\{\mathbf{a}\} \subseteq B_r(\mathbf{a}) \subseteq U$ so

$$U = \bigcup_{\mathbf{a}\in U} \{\mathbf{a}\} \subseteq \bigcup_{\mathbf{a}\in U} B_r(\mathbf{a}) \subseteq U.$$

Thus we must have equality throughout, in particular

$$U = \bigcup_{\mathbf{a}\in U} B_r(\mathbf{a})\,,$$

i.e. an open set U is the **union** of open balls.

Aside The open interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is an open set in $\mathbb{R}$ (which is why it is called an *open* interval). The general open set in $\mathbb{R}$ is a union of open intervals.

The box

$$\prod_{i=1}^{n}(a_i,b_i)=(a_1,b_1)\times(a_2,b_2)\times...\times(a_n,b_n)$$

is an open set in $\mathbb{R}^n$. Further, the general open set in $\mathbb{R}^n$, seen above as a union of open balls, can also be written as a union of open boxes.

End of Aside.

Definition 20 *Assume that $\mathbf{f}:U\subseteq\mathbb{R}^n\to\mathbb{R}^m$ is a function with domain U an open set. Then $\mathbf{f}$ is **continuous on** U if it is continuous at every point of U.*

Example 21

*i. Every **constant function** $c_{\mathbf{k}}:\mathbb{R}^n\to\mathbb{R}^m$ defined by $c_{\mathbf{k}}(\mathbf{x})=\mathbf{k}$ for all $\mathbf{x}\in\mathbb{R}^n$ (where $\mathbf{k}\in\mathbb{R}^m$), is continuous on $\mathbb{R}^n$.*

*ii. The **identity function** $I:\mathbb{R}^n\to\mathbb{R}^n$, defined by $I(\mathbf{x})=\mathbf{x}$ for all $\mathbf{x}\in\mathbb{R}^n$, is continuous on $\mathbb{R}^n$.*

Proof Follow immediately from the ε - δ definition, choose any $\delta>0$ in part a, and $\delta=\varepsilon$ in part b. ■

Example 22 *Let $\mathbf{c}\in\mathbb{R}^n$ and define $f:\mathbb{R}^n\to\mathbb{R}$ by $f(\mathbf{x})=\mathbf{c}\bullet\mathbf{x}$ for all $\mathbf{x}\in\mathbb{R}^n$. Then f is continuous on $\mathbb{R}^n$.*

Solution If $\mathbf{c}=\mathbf{0}$ the result is immediate. Assume $\mathbf{c}\neq\mathbf{0}$. Let $\mathbf{a}\in\mathbb{R}^n$ be given. Let $\varepsilon>0$ be given. Choose $\delta=\varepsilon/|\mathbf{c}|$. Assume $\mathbf{x}$ satisfies $0<|\mathbf{x}-\mathbf{a}|<\delta$. Then

$$\begin{aligned}|f(\mathbf{x})-f(\mathbf{a})| &= |\mathbf{c}\bullet\mathbf{x}-\mathbf{c}\bullet\mathbf{a}|=|\mathbf{c}\bullet(\mathbf{x}-\mathbf{a})|\\ &\leq |\mathbf{c}|\,|\mathbf{x}-\mathbf{a}|\quad\text{by Cauchy - Schwarz}\\ &< |\mathbf{c}|\,\delta=|\mathbf{c}|\,(\varepsilon/|\mathbf{c}|)\\ &= \varepsilon.\end{aligned}$$

Hence we have verified the definition of f is continuous at $\mathbf{a}$. Yet $\mathbf{a}$ was arbitrary so f is continuous on $\mathbb{R}^n$. ■

Questions about the continuity of *one* **vector-valued** function can always be reduced to questions about the continuity of *many* **scalar-valued** functions because of the following.

Proposition 23 *A function* $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$*, where* U *is an open set, is continuous at* $\mathbf{a} \in U$ *if, and only if, its real-valued component functions* $f^i : U \to \mathbb{R}$ *are continuous at* $\mathbf{a} \in U$ *for all* $1 \leq i \leq m$*.*

Proof Follows immediately from Proposition 7 : $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$ if, and only if, $\lim_{\mathbf{x}\to\mathbf{a}} f^i(\mathbf{x}) = f^i(\mathbf{a})$ for $1 \leq i \leq m$. ■

In the next example, knowing something of the vector-valued function tells us something about the coordinate functions.

Definition 24 *The* ***projection functions*** $p^i : \mathbb{R}^n \to \mathbb{R}$ *are defined by*

$$p^i(\mathbf{x}) = p^i\left(\begin{pmatrix} x^1 & x^2 & \cdots & \cdots & x^n \end{pmatrix}^T\right) = x^i$$

for $1 \leq i \leq n$

Note, rather confusingly, some authors use x^i in place of p^i and write $x^i(\mathbf{x}) = x^i$.

Corollary 25 *The projection functions are continuous on* $\mathbb{R}^n$*.*

Proof The identity function $I : \mathbb{R}^n \to \mathbb{R}^n$ can be written as

$$I(\mathbf{x}) = \mathbf{x} = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^n \end{pmatrix} = \begin{pmatrix} p^1(\mathbf{x}) \\ p^2(\mathbf{x}) \\ \vdots \\ p^n(\mathbf{x}) \end{pmatrix},$$

so the p^i are the coordinate functions of the identity function which we have seen above is continuous on $\mathbb{R}$. Hence result follows from Proposition 23. ■

The result also follows immediately from the ε - δ definition of continuity.

Example 26 *Let* $M \in M_{m,n}(\mathbb{R})$ *be a real valued* $m \times n$ *matrix and define* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ *by* $\mathbf{f}(\mathbf{x}) = M\mathbf{x}$ *for all* $\mathbf{x} \in \mathbb{R}^n$*. Then* $\mathbf{f}$ *is continuous on* $\mathbb{R}^n$*.*

Solution Write the matrix as rows:

$$M = \begin{pmatrix} \leftarrow & \mathbf{r}^1 & \rightarrow \\ \leftarrow & \mathbf{r}^2 & \rightarrow \\ & \vdots & \\ \leftarrow & \mathbf{r}^m & \rightarrow \end{pmatrix}.$$

Then matrix multiplication is

$$M\mathbf{x} = \begin{pmatrix} \mathbf{r}^1 \bullet \mathbf{x} \\ \mathbf{r}^2 \bullet \mathbf{x} \\ \vdots \\ \mathbf{r}^m \bullet \mathbf{x} \end{pmatrix}.$$

So the i-th component function is $\mathbf{x} \mapsto \mathbf{r}^i \bullet \mathbf{x}$ which we saw in Example 22 is continuous on $\mathbb{R}^n$. Hence by Proposition 23 the function $\mathbf{x} \mapsto M\mathbf{x}$ is continuous on $\mathbb{R}^n$. ■

1.9 Vector-valued Linear Functions of several variables

Some of the separate examples above can be combined for the functions in question are all examples of *linear functions.*

Definition 27 *A **linear function*** $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ *satisfies*

$$\mathbf{L}(\mathbf{a}+\mathbf{b}) = \mathbf{L}(\mathbf{a}) + \mathbf{L}(\mathbf{b}) \quad \text{and} \quad \mathbf{L}(\lambda\mathbf{a}) = \lambda\mathbf{L}(\mathbf{a})$$

for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$.

Note that $\mathbf{L}(\mathbf{0}) = \mathbf{L}(\mathbf{0}+\mathbf{0}) = \mathbf{L}(\mathbf{0}) + \mathbf{L}(\mathbf{0})$ and so $\mathbf{L}(\mathbf{0}) = \mathbf{0}$.

The functions

- $I : \mathbb{R}^n \to \mathbb{R}^n, \mathbf{x} \mapsto \mathbf{x}$,
- $\mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto \mathbf{c} \bullet \mathbf{x}$ for fixed $\mathbf{c} \in \mathbb{R}^n$,
- projection function $p^i : \mathbb{R}^n \to \mathbb{R}^n, \mathbf{x} \mapsto x^i$ and
- $\mathbb{R}^n \to \mathbb{R}^m, \mathbf{x} \mapsto M\mathbf{x}$ for fixed $M \in M_{m,n}(\mathbb{R})$

are all linear.

Example 28 *The function* $\mathbf{L} : \mathbb{R}^3 \to \mathbb{R}^2$ *given by*

$$\mathbf{L}(\mathbf{x}) = \begin{pmatrix} 2x+3y-z \\ 10x-5z \end{pmatrix},$$

for all $\mathbf{x} = (x, y, z)^T \in \mathbb{R}^3$ *is linear.*

The function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ *given by*

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x+1 \\ y+1 \end{pmatrix}$$

for all $\mathbf{x} = (x, y)^T \in \mathbb{R}^2$ *is not linear.*

Solution In the first part of this example we can write $\mathbf{L}(\mathbf{x})$ in matrix form:

$$\mathbf{L}(\mathbf{x}) = \begin{pmatrix} 2 & 3 & -1 \\ 10 & 0 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

In the second part we can simply note that $\mathbf{f}(\mathbf{0}) \neq \mathbf{0}$. ■

We have noted that if $M \in M_{m,n}(\mathbb{R})$ then $\mathbf{x} \mapsto M\mathbf{x}$ is a linear map $\mathbb{R}^n \to \mathbb{R}^m$. The next result shows that, conversely, **every** linear map is given by matrix multiplication. In the proof we need the

Observation If A is an $m \times n$ matrix then, for the usual basis vectors $\mathbf{e}_j \in \mathbb{R}^n$, $A\mathbf{e}_j = C_j$ the j-th column of A.

Lemma 29 *Assume $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. There exists a unique $m \times n$ matrix M such that $\mathbf{L}(\mathbf{x}) = M\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.*

Proof *Existence* Define M to be the $m \times n$ matrix

$$M = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{L}(\mathbf{e}_1) & \mathbf{L}(\mathbf{e}_2) & \dots & \mathbf{L}(\mathbf{e}_n) \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}, \tag{2}$$

where the i-th column is $\mathbf{L}(\mathbf{e}_i)$. In general, as noted above, if A is a matrix then $A\mathbf{e}_i = C_i$, the i-th column of A. Thus

$$M\mathbf{e}_i = \mathbf{L}(\mathbf{e}_i)$$

for all $1 \leq i \leq n$. This equality on basis vectors *extends by linearity* to all points of $\mathbb{R}^n$. To see what this means write a given $\mathbf{x} \in \mathbb{R}^n$ as

$$\mathbf{x} = \sum_{i=1}^{n} x^i \mathbf{e}_i,$$

with $x^i \in \mathbb{R}$. Then, since $\mathbf{L}$ is a linear map,

$$\begin{aligned} \mathbf{L}(\mathbf{x}) &= \mathbf{L}\left(\sum_{i=1}^{n} x^i \mathbf{e}_i\right) = \sum_{i=1}^{n} x^i \mathbf{L}(\mathbf{e}_i) \\ &= \sum_{i=1}^{n} x^i M\mathbf{e}_i, \quad \text{by the equality on basis vectors,} \\ &= M\left(\sum_{i=1}^{n} x^i \mathbf{e}_i\right) = M\mathbf{x}, \end{aligned}$$

as required.

Uniqueness Assume there exist two matrices with $\mathbf{L}(\mathbf{x}) = M\mathbf{x} = M'\mathbf{x}$. Then choosing $\mathbf{x} = \mathbf{e}_j$, the j-th usual basis vector in $\mathbb{R}^n$, we see $C_j = C'_j$, i.e. the columns of the matrices are the identical. True for all columns ($1 \le j \le n$) implies $M = M'$. ■

Stress We say that a linear function $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ is 'represented by' or 'is associated with' with the $m \times n$ matrix M.

If $m = 1$ then M is a $1 \times n$ matrix and it's transpose is a vector $\mathbf{c} = M^T$. In this case $L(\mathbf{x}) = \mathbf{c} \bullet \mathbf{x}$. This means we can recognise a linear (scalar valued) function of $\mathbf{x} = (x^1, ..., x^n)^T$; it is a linear combination of the x^i, it contains no cross-terms $x^i x^j$ and no powers $(x^i)^k$, $k \ge 2$.

Proposition 30 *If* $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ *is a linear map then it is continuous on* $\mathbb{R}^n$.

Proof In Example 26 it was shown that for a matrix $M \in M_{m,n}(\mathbb{R})$ the map $\mathbf{x} \to M\mathbf{x}$ is everywhere continuous on $\mathbb{R}^n$. The result then follows from Lemma 29. ■

1.10 Composition Laws

We have seen examples of Composition laws in Lemma's 13 and 16. The next Composition result concerns the situation

$$\mathbb{R}^p \xrightarrow{\mathbf{g}} \mathbb{R}^n \xrightarrow{\mathbf{f}} \mathbb{R}^m.$$

Theorem 31 *Given a function* $\mathbf{g} : A \subseteq \mathbb{R}^p \to \mathbb{R}^n$ *which is continuous at* $\mathbf{a} \in A$ *and a function* $\mathbf{f} : B \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *where* $\mathbf{g}(\mathbf{a}) \in B \subseteq \mathbb{R}^n$ *which is continuous at* $\mathbf{g}(\mathbf{a})$*, then the composition function* $\mathbf{f} \circ \mathbf{g} : A \to \mathbb{R}^m$ *is continuous at* $\mathbf{a}$.

Proof No new ideas; the proof is identical to that for the composition of two real-valued continuous functions of one variable, seen for example, in MATH20101. For this reason, no proof is given here. See Appendix. ■

We can rewrite Example 5 as saying

Example 32 *The* ***product function*** $p : \mathbb{R}^2 \to \mathbb{R}, (x, y)^T \mapsto xy$ *is continuous on* $\mathbb{R}^2$, We would like to state a similar result for the quotient function $(x,y)^T \mapsto x/y$. First, this is only defined for $y \neq 0$ so define $\mathbb{R}^\dagger = \mathbb{R}\setminus\{0\}$ and $q : \mathbb{R}\times\mathbb{R}^\dagger \to \mathbb{R}, (x,y)^T \mapsto x/y$.

We next look upon q as a quotient function

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{\mathbf{F}}{\mapsto} \begin{pmatrix} x \\ 1/y \end{pmatrix} \overset{p}{\mapsto} x\frac{1}{y} = \frac{x}{y},$$

so $q = \mathbf{F} \circ p$. To apply Theorem 31 we have to show that $\mathbf{F}$ is continuous on $\mathbb{R}\times\mathbb{R}^\dagger$. For this we have a lemma.

Lemma 33 *Assume $g : A \subseteq \mathbb{R} \to \mathbb{R}$ is continuous on the open set A. Then $\mathbf{G} : \mathbb{R} \times A \to \mathbb{R} \times \mathbb{R}$ given by*

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{\mathbf{G}}{\mapsto} \begin{pmatrix} x \\ g(y) \end{pmatrix}$$

is continuous on $\mathbb{R} \times A$.

Solution The function $\mathbf{G}$ can be written as

$$\mathbf{G}(\mathbf{x}) = \begin{pmatrix} p^1(\mathbf{x}) \\ g(p^2(\mathbf{x})) \end{pmatrix},$$

where p^1 and p^2 are the projection functions on $\mathbb{R}^2$. Then the first component of $\mathbf{G}$ is continuous on $\mathbb{R}^2$. The second component, $g \circ p^2$ is, by Theorem 31, continuous on $\mathbb{R} \times A$. Hence, by Proposition 23, $\mathbf{G}$ is continuous on $\mathbb{R} \times A$. ■

Note The same result would hold for $\mathbf{G} : A \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{\mathbf{G}}{\mapsto} \begin{pmatrix} g(x) \\ y \end{pmatrix}.$$

We can now combine all these results to state

Example 34 *The **quotient function** $q : \mathbb{R}\times\mathbb{R}^\dagger \to \mathbb{R}, (x,y)^T \mapsto x/y$ is continuous on $\mathbb{R}\times\mathbb{R}^\dagger$.*

The product function $p : \mathbb{R}^2 \to \mathbb{R} : (x,y)^T \to xy$ and the quotient function $q : \mathbb{R} \times \mathbb{R}^\dagger \to \mathbb{R} : (x,y)^T \to x/y$ are everywhere continuous

1.11 Continuity Laws

The following is the analogue of the same result for real-valued functions of one variable and I could have omitted it and it's proof by saying that there are no new ideas needed, just verify the ε - δ definition of continuity. But we can give alternative proofs using vector-valued functions.

Theorem 35 ***Laws for scalar-valued continuous functions*** *Assume that* $f, g : A \subseteq \mathbb{R}^n \to \mathbb{R}$ *are scalar-valued functions continuous at* $\mathbf{a} \in A$. *Then*

a. ***Sum Rule*** $f + g$ *is continuous at* $\mathbf{a}$;

b. ***Product Rule*** fg *is continuous at* $\mathbf{a}$;

c. ***Quotient Rule*** f/g *is continuous at* $\mathbf{a}$ *provided that* $g(\mathbf{a}) \neq 0$.

Proof by application of the Composite Rule. Define $\alpha : \mathbb{R}^2 \to \mathbb{R}$, $(x, y)^T \to x + y$. Then α is continuous on $\mathbb{R}^2$ (see problem sheet).

Define $\mathbf{F} : A \to \mathbb{R}^2$ by

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} f(\mathbf{x}) \\ g(\mathbf{x}) \end{pmatrix},$$

continuous at $\mathbf{a}$ by Proposition 23. Then

$$f + g = \alpha \circ \mathbf{F}, \quad fg = p \circ \mathbf{F} \quad \text{and} \quad \frac{f}{g} = q \circ \mathbf{F},$$

where p is the product function of Example 32 and q the quotient function of Example 34. The results then follow from Theorem 31. ■

Definition 36 *A* ***polynomial function*** $p : \mathbb{R}^n \to \mathbb{R}$ *is a sum of products of variables from* $\mathbb{R}^n$.

Instead of attempting to write the general polynomial function I illustrate with an example. In this example I will denote a constant function $\mathbf{x} \mapsto c$ by $c(\mathbf{x})$. So c represents both a scalar and a function. Importantly we know that constant functions are everywhere continuous.

Example 37 *An example of a polynomial on* $\mathbb{R}^3$ *would be*

$$\begin{aligned} p(\mathbf{x}) &= x^2 z + yz^3 + 3 = \left(p^1(\mathbf{x})\right)^2 p^3(\mathbf{x}) + p^2(\mathbf{x}) \left(p^3(\mathbf{x})\right)^3 + 3(\mathbf{x}) \\ &= \left(\left(p^1\right)^2 p^3 + p^2 \left(p^3\right)^3 + 3\right)(\mathbf{x}), \end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}^3$.

Now we see the use of projection functions. In this example, $p = (p^1)^2 p^3 + p^2 (p^3)^3 + 3$, a product and sum of functions continuous on $\mathbb{R}$, hence p is continuous on $\mathbb{R}$. This is true in general.

Corollary 38 *a) A polynomial function $p : \mathbb{R}^n \to \mathbb{R}$ given by a polynomial in the n variables is continuous on $\mathbb{R}^n$.*

b) A rational function r given by a ratio of polynomial functions $r = p/q$ on $\mathbb{R}^n$ is continuous at all points where it is defined (i.e. points $\mathbf{x} \in \mathbb{R}^n$ such that $q(\mathbf{x}) \neq 0$.)

Proof Immediate. ■

Note We can give Rules for Limits where we assume that the limits of f and g in Theorem 35 exist at $\mathbf{a}$. See Appendix.

Appendix for Section 1

1. Uniqueness of limit

Theorem 3 *Assume that* $\mathbf{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *is a function with domain* A *containing a deleted neighbourhood of* $\mathbf{a} \in \mathbb{R}^n$*. If* $\mathbf{f}(\mathbf{x})$ *has a limit as* $\mathbf{x} \to \mathbf{a}$ *then the limit is unique.*

I claim in the notes that this can be proved in exactly the same way as in the scalar-valued single variable case. So here I have taken that scalar-valued single variable found in the notes for MATH20101 and rewritten it in only a minor way.

Proof Assume that for the function $\mathbf{f}$ the limit is **not** unique. Let $\mathbf{b}_1 \neq \mathbf{b}_2$ be two of the different limit values (there may be more than two). Choose

$$\varepsilon = \frac{|\mathbf{b}_1 - \mathbf{b}_2|}{3} > 0.$$

From ε - δ definition of $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}_1$ we find $\delta_1 > 0$ such that $0 < |\mathbf{x} - \mathbf{a}| < \delta_1$ implies

$$|\mathbf{f}(\mathbf{x}) - \mathbf{b}_1| < \varepsilon. \tag{3}$$

Similarly, from the ε - δ definition of $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}_2$ we find $\delta_2 > 0$ such that $0 < |\mathbf{x} - \mathbf{a}| < \delta_2$ implies

$$|\mathbf{f}(\mathbf{x}) - \mathbf{b}_2| < \varepsilon. \tag{4}$$

Choose $\delta = \min(\delta_1, \delta_2) > 0$ and assume $\mathbf{x}_0$ satisfies $0 < |\mathbf{x}_0 - \mathbf{a}| < \delta$. For $\mathbf{x}_0$ both (3) and (4) hold. Hence

$$\begin{aligned}
|\mathbf{b}_1 - \mathbf{b}_2| &= |\mathbf{b}_1 - \mathbf{f}(\mathbf{x}_0) + \mathbf{f}(\mathbf{x}_0) - \mathbf{b}_2| \\
&\leq |\mathbf{b}_1 - \mathbf{f}(\mathbf{x}_0)| + |\mathbf{f}(\mathbf{x}_0) - \mathbf{b}_2| \\
&\qquad \text{by the triangle inequality,} \\
&< \varepsilon + \varepsilon \quad \text{by (3) and (4)}, \\
&= 2\varepsilon \\
&= \frac{2\,|\mathbf{b}_1 - \mathbf{b}_2|}{3}.
\end{aligned}$$

Dividing through by $|\mathbf{b}_1 - \mathbf{b}_2| \neq 0$ we get $1 < 2/3$, a contradiction. Hence the assumption is false and so, if it exists, $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x})$ is unique. ■

2. Example of limit

Example 6 *By verifying the* ε - δ *definition show that the vector-valued* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$, *given by*

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ xy \end{pmatrix},$$

has limit $(5,6)^T$ *at* $\mathbf{a} = (3,2)^T$.

Solution With $\mathbf{b} = (5,6)^T$ we saw in the notes the first equality in

$$\begin{aligned} |\mathbf{f}(\mathbf{x}) - \mathbf{b}|^2 &= \left| \begin{pmatrix} x+y-5 \\ xy-6 \end{pmatrix} \right|^2 = (x+y-5)^2 + (xy-6)^2 \\ &= \big((x-3) + (y-2) \big)^2 + \\ &\qquad + \big((x-3)(y-2) + 2(x-3) + 3(y-2) \big)^2 \end{aligned}$$

Then, by using the Triangle Inequality,

$$|\mathbf{f}(\mathbf{x}) - \mathbf{b}|^2 \le \big(|x-3| + |y-2| \big)^2 + \big(|x-3|\,|y-2| + 2\,|x-3| + 3\,|y-2| \big)^2$$

Assume $|\mathbf{x} - \mathbf{a}| < \delta$, when $|x-3| < \delta$ and $|y-2| < \delta$. Thus

$$|\mathbf{f}(\mathbf{x}) - \mathbf{b}|^2 \le (\delta + \delta)^2 + \left(\delta^2 + 5\delta\right)^2.$$

Assume $\delta \le 1$ to simplify this as

$$|\mathbf{f}(\mathbf{x}) - \mathbf{b}|^2 \le 40\delta^2.$$

Let $\varepsilon > 0$ be given. Assume $\delta = \min\left(1, \varepsilon/\sqrt{40}\right)$ and $0 < |\mathbf{x} - \mathbf{a}| < \delta$. For such $\mathbf{x}$ we have

$$|\mathbf{f}(\mathbf{x}) - \mathbf{b}| \le \sqrt{40}\delta \le \sqrt{40}\left(\frac{\varepsilon}{\sqrt{40}}\right) = \varepsilon.$$

Hence we have verified the ε - δ definition of $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}$. ■

3. Simplification of previous example

Remark If $\mathbf{x} \in \mathbb{R}^n$ then $|\mathbf{x}| \le \sum_{i=1}^n |x^i|$.

Proof Write

$$\mathbf{x} = \sum_{i=1}^n x^i \mathbf{e}_i,$$

where $\{\mathbf{e}_i\}_{1\le i\le n}$ is the standard basis of $\mathbb{R}^n$ and x^i are the coordinates of $\mathbf{x}$. Then

$$\begin{aligned}
|\mathbf{x}| &= \left|\sum_{i=1}^{n} x_i^i \mathbf{e}\right| \le \sum_{i=1}^{n} \left|x_i^i \mathbf{e}\right| \quad \text{by the triangle inequality,} \\
&= \sum_{i=1}^{n} \left|x^i\right| |\mathbf{e}_i| = \sum_{i=1}^{n} \left|x^i\right|.
\end{aligned}$$

Example 6 *By verifying the* ε - δ *definition show that the vector-valued* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$, *given by*

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+y \\ xy \end{pmatrix},$$

has limit $(5,6)^T$ *at* $\mathbf{a} = (3,2)^T$.

Solution With $\mathbf{b} = (5,6)^T$,

$$\begin{aligned}
|\mathbf{f}(\mathbf{x}) - \mathbf{b}| &= \left|\begin{pmatrix} x+y-5 \\ xy-6 \end{pmatrix}\right| \\
&\le |x+y-5| + |xy-6| \quad \text{by remark above,} \\
&= |(x-3)+(y-2)| + |(x-3)(y-2) + 2(x-3) + 3(y-2)| \\
&\le |x-3| + |y-2| + |x-3||y-2| + 2|x-3| + 3|y-2| \\
&\qquad \text{by the triangle inequality} \\
&\le \delta + \delta + \delta^2 + 2\delta + 3\delta \\
&\le 8\delta \quad \text{if } \delta \le 1.
\end{aligned}$$

So take $\delta = \min(1, \varepsilon/8)$. ■

4. Directional limit.

The directional limit in the direction $\mathbf{v} \neq \mathbf{0}$ was defined to be $\lim_{t\to 0+} \mathbf{f}(\mathbf{a} + t\mathbf{v})$. What does the left hand limit $\lim_{t\to 0-} \mathbf{f}(\mathbf{a} + t\mathbf{v})$ represent? Write $\eta = -t$ so $\eta \to 0+$ as $t \to 0-$. Then

$$\lim_{t\to 0-} \mathbf{f}(\mathbf{a} + t\mathbf{v}) = \lim_{\eta\to 0+} \mathbf{f}(\mathbf{a} + (-\eta)\mathbf{v}) = \lim_{\eta\to 0+} \mathbf{f}(\mathbf{a} + \eta(-\mathbf{v})),$$

which is the directional limit in the direction $-\mathbf{v}$.

5 Linear Functions are continuous

In the lectures we showed that linear maps are everywhere continuous by using the fact that the maps $\mathbf{x} \mapsto M\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$ and $M \in M_{m,n}(\mathbb{R})$ are everywhere continuous. Here we give an alternative proof. First

Lemma 39 *If $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map then there exists a positive constant C (depending on $\mathbf{L}$) such that $|\mathbf{L}(\mathbf{t})| \leq C|\mathbf{t}|$ for all $\mathbf{t} \in \mathbb{R}^n$.*

Proof Given $\mathbf{t} \in \mathbb{R}^n$ write it as $\mathbf{t} = \sum_{i=1}^n t_i^i \mathbf{e}$ where $\{\mathbf{e}_i\}$ is the usual basis of $\mathbb{R}^n$. Then, by linearity of $\mathbf{L}$,

$$\mathbf{L}(\mathbf{t}) = \sum_{i=1}^n t^i \mathbf{L}(\mathbf{e}_i).$$

Then, by Cauchy-Schwarz,

$$|\mathbf{L}(\mathbf{t})|^2 \leq \sum_{i=1}^n \left(t^i\right)^2 \sum_{i=1}^n \mathbf{L}(\mathbf{e}_i)^2 = |\mathbf{t}|^2 C^2,$$

if we choose

$$C = \left(\sum_{i=1}^n \mathbf{L}(\mathbf{e}_i)^2\right)^{1/2}.$$

■

Now our alternative proof that linear functions are continuous is immediate.

Proposition 30 *All linear maps $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ are everywhere continuous.*

Proof Let $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. So by previous result there exists $C > 0$ such that $|\mathbf{L}(\mathbf{t})| \leq C|\mathbf{t}|$ for all $\mathbf{t} \in \mathbb{R}^n$.

Let $\mathbf{a} \in \mathbb{R}^n$ be given. Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon/C$. Assume $\mathbf{x} \in \mathbb{R}^n$ satisfies $|\mathbf{x} - \mathbf{a}| < \delta$. Then

$$\begin{aligned} |L(\mathbf{x}) - L(\mathbf{a})| &= |L(\mathbf{x} - \mathbf{a})| \text{ since } L \text{ is linear,} \\ &\leq C|\mathbf{x} - \mathbf{a}| \\ &< C\delta = \varepsilon. \end{aligned}$$

Hence we have verified the $\delta - \varepsilon$ definition that f is continuous at $\mathbf{a}$. Since $\mathbf{a}$ was arbitrary f is everywhere continuous. ■

6 Linear Functions have directional limits

If $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ is linear and $\mathbf{a}, \mathbf{v} \in \mathbb{R}^n$ then

$$\lim_{t\to 0+} \mathbf{L}(\mathbf{a} + t\mathbf{v}) = \lim_{t\to 0+} (\mathbf{L}(\mathbf{a}) + t\mathbf{L}(\mathbf{v})) = \mathbf{L}(\mathbf{a}).$$

7. Composition Results for limits

In the lectures the following was unproved and left to the problem sheet.

Lemma 16 *Assume* $\mathbf{f} : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *where* A *contains a deleted neighbourhood of* $\mathbf{a} \in \mathbb{R}^n$. *Assume that* $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{c} \in \mathbb{R}^m$ *exists. Then, for any vector-valued function of one variable* $\mathbf{g} : (0, \eta) \to A \backslash \{\mathbf{a}\}$ *such that* $\lim_{t\to 0+} \mathbf{g}(t) = \mathbf{a}$, *we have*

$$\lim_{t\to 0+} \mathbf{f}(\mathbf{g}(t)) = \mathbf{c}.$$

This is called the limit of $\mathbf{f}$ *at* $\mathbf{a}$ *along* $\mathbf{g}$.

This result concerns the composition $\mathbf{f} \circ \mathbf{g}$ and the proof is simply a rewriting of the one for scalar-valued functions of one variable as seen, for example, in MATH20101. To get the proof to work examine the outer function, $\mathbf{f}$, first.

Proof Let $\varepsilon > 0$ be given. From the ε-δ definition that $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{c}$ there exists $\delta_1 > 0$ such that

$$0 < |\mathbf{x} - \mathbf{a}| < \delta_1 \implies |\mathbf{f}(\mathbf{x}) - \mathbf{c}| < \varepsilon. \tag{5}$$

Next choose $\varepsilon = \delta_1$ in the ε-δ definition of $\lim_{t\to 0+} \mathbf{g}(t) = \mathbf{a}$ to find $0 < \delta_2 < \eta$ such that if $0 < t < \delta_2$ then $|\mathbf{g}(t) - \mathbf{a}| < \delta_1$.

Note that in the statement of the result we are assuming $\mathbf{g} : (0, \eta) \to \mathbb{R}^n \backslash \{\mathbf{a}\}$, in particular, $\mathbf{g}(t) \neq \mathbf{a}$ for any $0 < t < \eta$. Write this as $0 < |\mathbf{g}(t) - \mathbf{a}|$. Then $0 < t < \delta_2$ implies $0 < |\mathbf{g}(t) - \mathbf{a}| < \delta_1$. Hence, by (5) with $\mathbf{x} = \mathbf{g}(t)$, we find that $|\mathbf{f}(\mathbf{g}(t)) - \mathbf{c}| < \varepsilon$.

That is, we have shown that for all $\varepsilon > 0$, there exists $\delta_2 > 0$ such that $0 < t < \delta_2$ implies $|\mathbf{f}(\mathbf{g}(t)) - \ell| < \varepsilon$. This is the ε-δ definition of $\lim_{t\to 0+} \mathbf{f}(\mathbf{g}(t)) = \mathbf{c}$. ■

This result was used in the notes to prove

Example 17 *Show that*

$$f(\mathbf{x}) = \begin{cases} 1 \text{ if } \mathbf{x} = (x, x^2), \ x \neq 0 \\ 0 \text{ otherwise} \end{cases}$$

has no limit at $\mathbf{x} = \mathbf{0}$.

We prove this from the definition of the limit.

Solution We will show that the negation of the definition of the limit holds. The definition states

$$\exists b \in \mathbb{R}, \forall \varepsilon > 0, \ \exists \delta > 0 : \forall \mathbf{x} \in \mathbb{R}^n, \ 0 < |\mathbf{x} - \mathbf{a}| < \delta \implies |f(\mathbf{x}) - b| < \varepsilon.$$

The negation states

$$\forall b \in \mathbb{R}, \exists \varepsilon > 0, \ \forall \delta > 0 : \exists \mathbf{x} \in \mathbb{R}^n, \ 0 < |\mathbf{x} - \mathbf{a}| < \delta \text{ and } |f(\mathbf{x}) - b| \geq \varepsilon. \quad (6)$$

Let $b \in \mathbb{R}$ be given. In any disc about the origin (i.e. for any $\delta > 0$) we can find points $\mathbf{x}_0 : f(\mathbf{x}_0) = 0$ and $\mathbf{x}_1 : f(\mathbf{x}_1) = 1$. Then

$$\begin{aligned} 1 &= |f(\mathbf{x}_1) - f(\mathbf{x}_0)| = |(f(\mathbf{x}_1) - b) - (f(\mathbf{x}_0) - b)| \\ &\leq |f(\mathbf{x}_1) - b| + |f(\mathbf{x}_0) - b| \end{aligned}$$

by the triangle inequality. This means that at least one of $|f(\mathbf{x}_1) - b|$ and $|f(\mathbf{x}_0) - b|$ must be $\geq 1/2$. Hence we can choose $\varepsilon = 1/2$ and (6) will hold.■

Hopefully this is sufficiently logically complicated to illustrate the virtue of looking at limits along curves.

If Lemma 16 concerned the situation

$$\mathbb{R} \xrightarrow{\mathbf{g}} \mathbb{R}^n \xrightarrow{\mathbf{f}} \mathbb{R}^m,$$

we can next deal with the more general

$$\mathbb{R}^p \xrightarrow{\mathbf{g}} \mathbb{R}^n \xrightarrow{\mathbf{f}} \mathbb{R}^m.$$

To be consistent with occurrences of composition later in the course we will write the vectors in $\mathbb{R}^p$ as $\mathbf{x}$ and in $\mathbb{R}^n$ as $\mathbf{y}$ (unfortunately this is not consistent with Lemma 16).

In the previous result we assumed $\mathbf{f}$ had a limit at $\mathbf{a}$, but not necessarily defined at $\mathbf{a}$. For this reason we had to omit $\mathbf{a}$ from the image of $\mathbf{g}$. In the following result we simplify matters by assuming $\mathbf{f}$ is continuous, it is then necessarily defined at all points.

Theorem 40 Assume that $\mathbf{g} : \mathbb{R}^p \to \mathbb{R}^n$ has a limit at $\mathbf{a} \in \mathbb{R}^p$ and write $\mathbf{b} = \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x})$. Assume that $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is continuous in an open neighbourhood of $\mathbf{b}$. Then

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{g}(\mathbf{x})) = \mathbf{f}(\mathbf{b}).$$

Alternatively the conclusion can be written as

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{g}(\mathbf{x})) = \mathbf{f}\left(\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x})\right).$$

Proof Let $\varepsilon > 0$ be given. From the ε-δ definition that $\mathbf{f}$ is continuous at $\mathbf{b}$ there exists $\delta_1 > 0$ such that

$$|\mathbf{y} - \mathbf{b}| < \delta_1 \implies |\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{b})| < \varepsilon. \tag{7}$$

Next choose $\varepsilon = \delta_1$ in the ε-δ definition of $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x}) = \mathbf{b}$ to find $\delta_2 > 0$ such that if $0 < |\mathbf{x} - \mathbf{a}| < \delta_2$ then $|\mathbf{g}(\mathbf{x}) - \mathbf{b}| < \delta_1$. Then by (7) with $\mathbf{y} = \mathbf{g}(\mathbf{x})$ we find that $|\mathbf{f}(\mathbf{g}(\mathbf{x})) - \mathbf{f}(\mathbf{b})| < \varepsilon$.

That is, we have shown that for all $\varepsilon > 0$, there exists $\delta_2 > 0$ such that $0 < |\mathbf{x} - \mathbf{a}| < \delta_2$ implies $|\mathbf{f}(\mathbf{g}(\mathbf{x})) - \mathbf{f}(\mathbf{b})| < \varepsilon$. This is the ε-δ definition of $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{g}(\mathbf{x})) = \mathbf{f}(\mathbf{b})$.

■

If we assume further that $\mathbf{g}$ is continuous at $\mathbf{a}$ then we get

$$\begin{aligned}\lim_{\mathbf{x}\to\mathbf{a}} (\mathbf{f} \circ \mathbf{g})(\mathbf{x}) &= \lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{g}(\mathbf{x})) = \mathbf{f}\left(\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{g}(\mathbf{x})\right) \\ &= \mathbf{f}(\mathbf{g}(\mathbf{a})) = (\mathbf{f} \circ \mathbf{g})(\mathbf{a}).\end{aligned}$$

That is, $\mathbf{f} \circ \mathbf{g}$ is continuous at $\mathbf{a}$. This result was stated in the lectures, without proof, as

Theorem 31 *Given a function* $\mathbf{g} : U \subseteq \mathbb{R}^p \to \mathbb{R}^n$ *which is continuous at* $\mathbf{a} \in U$ *and a function* $\mathbf{f} : V \subseteq \mathbb{R}^n \to \mathbb{R}^m$*, where* $\mathbf{g}(\mathbf{a}) \in V$*, which is continuous at* $\mathbf{g}(\mathbf{a})$*, then the composition function* $\mathbf{f} \circ \mathbf{g} : U \to \mathbb{R}^m$ *is continuous at* $\mathbf{a}$.

A special case of this is when $\mathbf{g} : \mathbb{R} \to \mathbb{R}^n$, $t \mapsto \mathbf{a} + t\mathbf{v}$ for some $\mathbf{a}, \mathbf{v} \in \mathbb{R}^n$.

Proposition 41 *If* $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ *is continuous at* $\mathbf{a} \in \mathbb{R}^n$ *then, for all* $\mathbf{v} \in \mathbb{R}^n$*,* $\mathbf{f}(\mathbf{a} + t\mathbf{v})$ *is continuous at* $t = 0$.

The converse is not true. Even if $\mathbf{f}(\mathbf{a}+t\mathbf{v})$ is continuous at $t=0$ for all $\mathbf{v}\in\mathbb{R}^n$, it may be that $\lim_{\mathbf{x}\to\mathbf{a}}\mathbf{f}(\mathbf{x})$ does not exist. And if the limit were to exist it may not equal $\mathbf{f}(\mathbf{a})$, i.e. $\mathbf{f}$ might not be continuous at $\mathbf{a}$.

That is

$$\boxed{\begin{array}{l} \mathbf{f} \text{ continuous at } \mathbf{a} \implies \forall\,\mathbf{v}\in\mathbb{R}^n, \mathbf{f}\left(\mathbf{a}+t\mathbf{v}\right) \text{ is continuous at } t=0, \\ \forall\,\mathbf{v}\in\mathbb{R}^n, \mathbf{f}\left(\mathbf{a}+t\mathbf{v}\right) \text{ is continuous at } t=0 \not\Longrightarrow \mathbf{f} \text{ continuous at } \mathbf{a}. \end{array}}$$

8. Limit Laws

Though Proposition 7 reduces the verification of limits for vector-valued functions to scalar-valued functions, can the verification for scalar-valued functions be simplified?

One way is to build a function from ‘simpler’ functions or, conversely break down a given ‘complicated’ function into ‘simpler’ ones. Since we can only multiply and divide scalars not vectors, the first result concerns *scalar*-valued functions.

Theorem 42 ***Limit Laws for Scalar-Valued functions*** *Assume that $f,g:A\subseteq\mathbb{R}^n\to\mathbb{R}$ are **scalar-valued** functions with domain A containing a deleted neighbourhood of $\mathbf{a}\in\mathbb{R}^n$. If $\lim_{\mathbf{x}\to\mathbf{a}}f(\mathbf{x})=b\in\mathbb{R}$ and $\lim_{\mathbf{x}\to\mathbf{a}}g(\mathbf{x})=c\in\mathbb{R}$ then*

a) ***Sum Rule*** $\lim_{\mathbf{x}\to\mathbf{a}}\left(f(\mathbf{x})+g(\mathbf{x})\right)=b+c$;

b) ***Product Rule*** $\lim_{\mathbf{x}\to\mathbf{a}}\left(f(\mathbf{x})\,g(\mathbf{x})\right)=bc$;

c) ***Quotient Rule*** $\lim_{\mathbf{x}\to\mathbf{a}}\left(\dfrac{f(\mathbf{x})}{g(\mathbf{x})}\right)=\dfrac{b}{c}$, *provided that $c\neq 0$.*

Proof Use the method seen in the proof of Theorem 35. Define $\alpha:\mathbb{R}^2\to\mathbb{R}$, $(x,y)^T\to x+y$. Then α is continuous on $\mathbb{R}^2$ (see problem sheet). Define $\mathbf{F}:A\subseteq\mathbb{R}^n\to\mathbb{R}^2$ by

$$\mathbf{F}(\mathbf{x})=\begin{pmatrix} f(\mathbf{x}) \\ g(\mathbf{x}) \end{pmatrix},$$

for which

$$\lim_{\mathbf{x}\to\mathbf{a}}\mathbf{F}(\mathbf{x})=\begin{pmatrix} \lim_{\mathbf{x}\to\mathbf{a}}f(\mathbf{x}) \\ \lim_{\mathbf{x}\to\mathbf{a}}g(\mathbf{x}) \end{pmatrix}$$

exists by assumption.

Then

$$f+g=\alpha\circ\mathbf{F},\quad fg=p\circ\mathbf{F}\quad\text{and}\quad\frac{f}{g}=q\circ\mathbf{F},$$

where p is the product function of Example 32 and q the quotient function of Example 34. The results then follow from Theorem 40. $\blacksquare$

For *vector* - valued functions we have

Theorem 43 ***Limit Laws for Vector-Valued functions*** *Assume that* $\mathbf{f},\mathbf{g}:A\subseteq\mathbb{R}^n\to\mathbb{R}^m$ *are* ***vector-valued*** *functions while* $h:A\subseteq\mathbb{R}^n\to\mathbb{R}$ *is scalar valued. Assume the domain* A *contains a deleted neighbourhood of* $\mathbf{a}\in\mathbb{R}^n$*. If* $\lim_{\mathbf{x}\to\mathbf{a}}\mathbf{f}(\mathbf{x})=\mathbf{b}\in\mathbb{R}^m$, $\lim_{\mathbf{x}\to\mathbf{a}}\mathbf{g}(\mathbf{x})=\mathbf{c}\in\mathbb{R}^m$ *and* $\lim_{\mathbf{x}\to\mathbf{a}}h(\mathbf{x})=\ell$ *then*

a) $\lim_{\mathbf{x}\to\mathbf{a}}(\mathbf{f}(\mathbf{x})+\mathbf{g}(\mathbf{x}))=\mathbf{b}+\mathbf{c}$;

b) $\lim_{\mathbf{x}\to\mathbf{a}}(h(\mathbf{x})\,\mathbf{f}(\mathbf{x}))=\ell\mathbf{b}$;

c) $\lim_{\mathbf{x}\to\mathbf{a}}(\mathbf{f}(\mathbf{x})\bullet\mathbf{g}(\mathbf{x}))=\mathbf{b}\bullet\mathbf{c}$.

Proof By Proposition 7 parts a. and b. are equivalent to $\lim_{\mathbf{x}\to\mathbf{a}}(f^i(\mathbf{x})+g^i(\mathbf{x}))=b^i+c^i$ and $\lim_{\mathbf{x}\to\mathbf{a}}h(x)\,f^i(\mathbf{x})=\ell b^i$ for $1\le i\le n$. Yet these latter results, on scalar-valued functions alone, follow from Theorem 42.

Part c. has been left to the Problem Sheet. $\blacksquare$